WP11 Solutions

185. $\int_0^1\int_1^2\int_2^3 (x^2+\ln y+z)\,dx\,dy\,dz$

$= \int_1^2\int_2^3\int_0^1 (x^2+\ln y+z)\,dz\,dx\,dy$

$= \int_1^2\int_2^3 \left(zx^2+z\ln y+\frac{1}{2}z^2\right|_0^1 dx\,dy$

$= \int_1^2\int_2^3 (x^2+\ln y+\frac{1}{2})\,dx\,dy$

$= \int_1^2 \left(\frac{1}{3}x^3+x\ln y+\frac{1}{2}x\right|_2^3 dy$

$= \int_1^2 \left[(9+3\ln y+\frac{3}{2})-(\frac{8}{3}+2\ln y+1)\right] dy$

$= \int_1^2 \left(\frac{19}{3}+\ln y+\frac{5}{2}\right) dy$

$= \left(\frac{19}{3}y+y\ln y-y+\frac{5}{2}y\right|_1^2$

$= \left(\frac{38}{3}+2\ln 2-2+5\right)-\left(\frac{19}{3}+0-1+\frac{5}{2}\right)$

$= \frac{47}{6}+2\ln 2$

195. $\iiint_E (x+2yz)\,dV;\ E=[0,1]\times[0,x]\times[0,5-x-y]$

$= \int_0^1\int_0^x\int_0^{5-x-y} (x+2yz)\,dz\,dy\,dx$

$= \int_0^1\int_0^x \left(zx+yz^2\right|_0^{5-x-y} dy\,dx$

$= \int_0^1\int_0^x (5x-x^2+25y-11xy-10y^2+x^2y+2xy^2+y^3)\,dy\,dx$

$= \int_0^1 \left(5xy-x^2y+\frac{25}{2}y^2-\frac{11}{2}xy^2-\frac{10}{3}y^3+\frac{1}{2}x^2y^2+\frac{2}{3}xy^3+\frac{1}{4}y^4\right|_0^x dx$

$= \int_0^1 \left(5x^2-x^3+\frac{25}{2}x^2-\frac{11}{2}x^3-\frac{10}{3}x^3+\frac{1}{2}x^4+\frac{2}{3}x^4+\frac{1}{4}x^4\right) dx$

$= \int_0^1 \left(\frac{35}{2}x^2-\frac{59}{6}x^3+\frac{17}{12}x^4\right) dx$

$= \left(\frac{35}{6}x^3-\frac{59}{24}x^4+\frac{17}{60}x^5\right|_0^1$

$= \frac{700}{120}-\frac{295}{120}+\frac{34}{120}$

$= \frac{439}{120}$

222. $f(x,y,z)=xyz \quad E=[0,1]\times[0,1]\times[0,1]$

$f_{avg} = \frac{1}{\int_0^1\int_0^1\int_0^1 dz\,dy\,dx}\int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx$

$= \frac{1}{\int_0^1\int_0^1 z|_0^1\,dy\,dx}\int_0^1\int_0^1 \frac{1}{2}xyz^2\Big|_0^1 dy\,dx$

$= \frac{1}{\int_0^1\int_0^1 dy\,dx}\int_0^1\int_0^1 \frac{1}{2}xy\,dy\,dx$

$= \frac{1}{\int_0^1 y|_0^1\,dx}\int_0^1 \frac{1}{4}xy^2\Big|_0^1 dx$

$= \frac{1}{\int_0^1 dx}\int_0^1 \frac{1}{4}x\,dx$

$= \frac{1}{x|_0^1}\left(\frac{1}{8}x^2\right|_0^1$

$= \frac{1}{8}$

236. $x+y+z=5,\ T(x,y,z)=x+y+xy$

$T_{avg} = \frac{1}{\int_0^5\int_0^{5-x}\int_0^{5-x-y} dz\,dy\,dx}\int_0^5\int_0^{5-x}\int_0^{5-x-y} (x+y+xy)\,dz\,dy\,dx$

Note: $\int_0^5\int_0^{5-x}\int_0^{5-x-y} dz\,dy\,dx = \frac{125}{6}$

AND

$\int_0^5\int_0^{5-x}\int_0^{5-x-y} (x+y+xy)\,dz\,dy\,dx$

$= \int_0^5\int_0^{5-x} z(x+y+xy)\Big|_0^{5-x-y} dy\,dx$

$= \int_0^5\int_0^{5-x} (5x-x^2-3xy+5y+y^2-x^2y-xy^2)\,dy\,dx$

$= \int_0^5 \left(5xy-x^2y-\frac{3}{2}xy^2+\frac{5}{2}y^2+\frac{1}{3}y^3-\frac{1}{2}x^2y^2-\frac{1}{3}xy^3\right|_0^{5-x} dx$

$= \int_0^5 \left(\frac{1}{12}x^4-\frac{5}{6}x^3+25x^2-\frac{625}{6}x+\frac{625}{6}\right) dx$

$= \left(\frac{1}{60}x^5-\frac{5}{24}x^4+\frac{25}{3}x^3-\frac{625}{12}x^2+\frac{625}{6}x\right|_0^5$

$= \frac{71875}{120}$

So $T_{avg} = \frac{115}{4} = 28.75^\circ$

241. $B=\{(x,y,z) \mid x^2+y^2\le 9, x\ge 0, y\ge 0, 0\le z\le 1\}$
$f(x,y,z)=z$

$$\int_0^1\int_0^{\pi/2}\int_0^3 zr\,dr\,d\theta\,dz$$
$$=\int_0^1\int_0^{\pi/2} \tfrac{1}{2}r^2 z\Big|_0^3 d\theta\,dz$$
$$=\int_0^1\int_0^{\pi/2} \tfrac{9}{2}z\,d\theta\,dz$$
$$=\int_0^1 \tfrac{9}{2}z\theta\Big|_0^{\pi/2} dz$$
$$=\int_0^1 \tfrac{9\pi}{4}z\,dz$$
$$=\tfrac{9\pi}{4}\left(\tfrac{1}{2}z^2\right)\Big|_0^1$$
$$=9\pi/8$$

250. E = right circular cylinder $r=\cos\theta$ the $r\theta$-plane and $r^2+z^2=9$

a) $E=\{(r,\theta,z) \mid 0\le\theta\le\pi, 0\le r\le\cos\theta, 0\le z\le\sqrt{9-r^2}\}$

b) $\displaystyle\int_0^{\pi}\int_0^{\cos\theta}\int_0^{\sqrt{9-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta$

260. E above xy plane, below $z=1$ outside $x^2+y^2-z^2=1$ inside $x^2+y^2=2$

$$V=\int_0^{2\pi}\int_1^2\int_0^1 r\,dz\,dr\,d\theta$$
$$=\int_0^{2\pi}\int_1^2 zr\Big|_{z=0}^1 dr\,d\theta$$
$$=\int_0^{2\pi}\int_1^2 r\,dr\,d\theta$$
$$=\int_0^{2\pi} \tfrac{1}{2}r^2\Big|_1^2 d\theta$$
$$=\int_0^{2\pi}(2-\tfrac{1}{2})\,d\theta$$
$$=\int_0^{2\pi}\tfrac{3}{2}d\theta$$
$$=(\tfrac{3}{2}\theta\Big|_0^{2\pi}$$
$$V=3\pi$$

262. E inside $x^2+y^2+z^2=1$; above xy plane inside $z=\sqrt{x^2+y^2}$

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi/4}\tfrac{1}{3}\rho^3\sin\phi\Big|_0^1 d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^1 \tfrac{1}{3}\sin\phi\,d\phi\,d\theta$$
$$=\int_0^{2\pi}(-\tfrac{1}{3}\cos\phi\Big|_0^{\pi/4} d\theta$$
$$=\int_0^{2\pi}\left(-\tfrac{\sqrt{2}}{6}+\tfrac{1}{3}\right)d\theta$$
$$=\tfrac{2-\sqrt{2}}{6}\theta\Big|_0^{2\pi}$$
$$V=\tfrac{2-\sqrt{2}}{3}\pi$$

295. $q(x,y,z)=k\sqrt{x^2+y^2+z^2}\ \frac{\mu C}{cm^3}$ and $Q=\iiint_B q(x,y,z)\,dV$

$$Q=\int_0^{2\pi}\int_0^{\pi}\int_0^r k\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi}\tfrac{k}{4}\rho^4\sin\phi\Big|_0^r d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi}\tfrac{k}{4}r^4\sin\phi\,d\phi\,d\theta$$
$$=\int_0^{2\pi}(-\tfrac{k}{4}r^4\cos\phi\Big|_0^{\pi} d\theta$$

$$Q=\int_0^{2\pi}\left[\tfrac{k}{4}r^4-(-\tfrac{k}{4}r^4)\right]d\theta$$
$$=\int_0^{2\pi}\tfrac{k}{2}r^4\,d\theta$$
$$=\tfrac{k}{2}r^4\theta\Big|_0^{2\pi}$$
$$=kr^4\pi\ \mu C$$