WP 12 Solutions

297. R: triangle (0,0), (0,3), (6,0)

$\rho(x,y) = xy$

$$m = \iint_R xy\,dA$$

$$m = \int_0^6 \int_0^{3-\frac{1}{2}x} xy\,dy\,dx$$

$$= \int_0^6 \frac{1}{2}xy^2 \Big|_0^{3-\frac{1}{2}x} dx$$

$$= \int_0^6 \frac{1}{2}x\left(3-\frac{1}{2}x\right)^2 dx$$

$$= \int_0^6 \left(\frac{1}{8}x^3 - \frac{3}{2}x^2 + \frac{9}{2}x\right)dx$$

$$= \left(\frac{1}{32}x^4 - \frac{1}{2}x^3 + \frac{9}{4}x^2\right)\Big|_0^6$$

$$= \left(\frac{81}{2} - 108 + 81\right) - 0$$

$$m = 27/2$$

302. R: trapezoid $y=0, y=1, y=x$

$y=-x+3$; $\rho(x,y) = 2x+y$

$$m = \iint_R (2x+y)\,dA$$

$$m = \int_0^1\int_0^x (2x+y)\,dy\,dx + \int_1^2\int_0^1 (2x+y)\,dy\,dx + \int_2^3\int_0^{3-x} (2x+y)\,dy\,dx$$

$$= \int_0^1 \left(2xy+\frac{1}{2}y^2\Big|_0^x\right)dx + \int_1^2 \left(2xy+\frac{1}{2}y^2\Big|_0^1\right) + \int_2^3 \left(2xy+\frac{1}{2}y^2\Big|_0^{3-x}\right)dx$$

$$= \int_0^1 \frac{5}{2}x^2\,dx + \int_1^2 \left(2x+\frac{1}{2}\right)dx + \int_2^3 \left(-\frac{3}{2}x^2+3x+\frac{9}{2}\right)dx$$

$$= \left(\frac{5}{6}x^3\Big|_0^1\right) + \left(x^2+\frac{1}{2}x\Big|_1^2\right) + \left(-\frac{1}{2}x^3+\frac{3}{2}x^2+\frac{9}{2}x\Big|_2^3\right)$$

$$= \left(\frac{5}{6}-0\right)+(4+1)-\left(1+\frac{1}{2}\right)+\left(-\frac{27}{2}+\frac{27}{2}+\frac{27}{2}\right)-(-4+6+9)$$

$$= \frac{5}{6}+5-\frac{3}{2}+\frac{27}{2}-11$$

$$m = 41/6$$

309. R: triangle (0,0) (0,3) (6,0)

$\rho(x,y) = xy$

From 297: $m = \frac{27}{2}$

a) $$M_x = \iint_R y(xy)\,dA = \int_0^6\int_0^{3-\frac{1}{2}x} xy^2\,dy\,dx$$

$$= \int_0^6 \frac{1}{3}xy^3 \Big|_0^{3-\frac{1}{2}x} dx$$

$$= \int_0^6 \left(-\frac{1}{24}x^4 + \frac{3}{4}x^3 - \frac{9}{2}x^2 + 9x\right)dx$$

$$= -\frac{1}{120}x^5 + \frac{3}{16}x^4 - \frac{3}{2}x^3 + \frac{9}{2}x^2\Big|_0^6$$

$$= -\frac{7776}{120} + \frac{3888}{16} - \frac{648}{2} + \frac{324}{2}$$

$$= \frac{-15552+58320-77760+38880}{240}$$

$$M_x = 81/5$$

$$M_y = \iint_R x(xy)\,dA = \int_0^6\int_0^{3-\frac{1}{2}x} x^2y\,dy\,dx$$

$$= \int_0^6 \frac{1}{2}x^2y^2 \Big|_0^{3-\frac{1}{2}x} dx$$

$$= \int_0^6 \left(\frac{1}{8}x^4 - \frac{3}{2}x^3 + \frac{9}{2}x^2\right)dx$$

$$= \frac{1}{40}x^5 - \frac{3}{8}x^4 + \frac{3}{2}x^3\Big|_0^6$$

$$= \frac{7776}{40} - \frac{3888}{8} + \frac{648}{2}$$

$$= \frac{7776-19440+12960}{40}$$

$$M_y = 162/5$$

b) $\bar{x} = \frac{M_y}{m}$ $\qquad \bar{y} = \frac{M_x}{m}$

$\bar{x} = \frac{162/5}{27/2}$ $\qquad \bar{y} = \frac{81/5}{27/2}$

$\bar{x} = 12/5$; $\bar{y} = 6/5$

314. R: trapezoid $y=0$, $y=1$, $y=x$, $y=-x+3$; $\rho(x,y)=2x+y$

From 302: $m = 41/6$

a) $M_x = \iint_R y(2x+y)\,dA$

$M_x = \int_0^1\int_0^x (2xy+y^2)\,dy\,dx + \int_1^2\int_0^1 (2xy+y^2)\,dy\,dx + \int_2^3\int_0^{3-x} (2xy+y^2)\,dy\,dx$

$= \int_0^1 \left(xy^2+\frac{1}{3}y^3\right)\Big|_0^x dx + \int_1^2 \left(xy^2+\frac{1}{3}y^3\right)\Big|_0^1 dx + \int_2^3 \left(xy^2+\frac{1}{3}y^3\right)\Big|_0^{3-x} dx$

$= \int_0^1 \frac{4}{3}x^3\,dx + \int_1^2 \left(x+\frac{1}{3}\right)dx + \int_2^3 \left(\frac{2}{3}x^3-3x^2+9\right)dx$

$= \left(\frac{1}{3}x^4\Big|_0^1 + \left(\frac{1}{2}x^2+\frac{1}{3}x\Big|_1^2 + \left(\frac{1}{6}x^4-x^3+9x\Big|_2^3\right.\right.\right.$

$= \left(\frac{1}{3}-0\right) + \left(2+\frac{2}{3}\right) - \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{27}{2}-27+27\right) - \left(\frac{8}{3}-8+18\right)$

$M_x = 3$

$M_y = \iint_R x(2x+y)\,dA$

$M_y = \int_0^1\int_0^x (2x^2+xy)\,dy\,dx + \int_1^2\int_0^1 (2x^2+xy)\,dy\,dx + \int_2^3\int_0^{3-x} (2x^2+xy)\,dy\,dx$

$= \int_0^1 \left(2x^2y+\frac{1}{2}xy^2\right)\Big|_0^x dx + \int_1^2 \left(2x^2y+\frac{1}{2}xy^2\right)\Big|_0^1 dx + \int_2^3 \left(2x^2y+\frac{1}{2}xy^2\right)\Big|_0^{3-x} dx$

$= \int_0^1 \frac{5}{2}x^3\,dx + \int_1^2 \left(2x^2+\frac{1}{2}x\right)dx + \int_2^3 \left(-\frac{3}{2}x^3+3x^2+\frac{9}{2}x\right)dx$

$= \left(\frac{5}{8}x^4\Big|_0^1 + \left(\frac{2}{3}x^3+\frac{1}{4}x^2\Big|_1^2 + \left(-\frac{3}{8}x^4+x^3+\frac{9}{4}x^2\Big|_2^3\right.\right.\right.$

$= \left(\frac{5}{8}-0\right) + \left(\frac{16}{3}+1\right) - \left(\frac{2}{3}+\frac{1}{4}\right) + \left(-\frac{243}{8}+27+\frac{81}{4}\right) - (-6+8+9)$

$M_y = 143/12$

b) $\bar{x} = \frac{M_y}{m} = \frac{\frac{143}{12}}{41/6}$ $\qquad \bar{y} = \frac{M_x}{m} = \frac{3}{41/6}$

$\bar{x} = 143/82 \qquad \bar{y} = 18/41$

321. R: triangle $(0,0),(0,3),(6,0)$ $\rho(x,y)=xy$

a) $I_x=\iint_R y^2(xy)\,dA$

$=\int_0^6\int_0^{-\frac{1}{2}x+3} xy^3\,dy\,dx$

$=\int_0^6 \frac{1}{4}xy^4\Big|_0^{-\frac{1}{2}x+3}dx$

$=\int_0^6\left(\frac{1}{64}x^5-\frac{3}{8}x^4+\frac{27}{8}x^3-\frac{27}{2}x^2+\frac{81}{4}x\right)dx$

$=\left(\frac{1}{384}x^6-\frac{3}{40}x^5+\frac{27}{32}x^4-\frac{9}{2}x^3+\frac{81}{8}x^2\right|_0^6$

$=\frac{46656}{384}-\frac{23328}{40}+\frac{34992}{32}-\frac{1944}{2}+\frac{2916}{8}$

$I_x=\frac{243}{10}$

$I_y=\iint_R x^2(xy)\,dA$

$=\int_0^6\int_0^{-\frac{1}{2}x+3} x^3y\,dy\,dx$

$=\int_0^6 \frac{1}{2}x^3y^2\Big|_0^{-\frac{1}{2}x+3}dx$

$=\int_0^6\left(\frac{1}{8}x^5-\frac{3}{2}x^4+\frac{9}{2}x^3\right)dx$

$=\left(\frac{1}{48}x^6-\frac{3}{10}x^5+\frac{9}{8}x^4\right|_0^6$

$=\frac{46656}{48}-\frac{23328}{10}+\frac{11664}{8}$

$I_y=\frac{486}{5}$

$I_0=I_x+I_y$

$=\frac{243}{10}+\frac{486}{5}$

$=\frac{1215}{10}$

$I_0=\frac{243}{2}$

b) $R_x=\sqrt{\dfrac{243/10}{27/2}}$ $\quad R_y=\sqrt{\dfrac{486/5}{27/2}}$ $\quad R_0=\sqrt{\dfrac{243/2}{27/2}}$

$R_x=\frac{3\sqrt{5}}{5}$ $\quad R_y=\frac{6\sqrt{5}}{5}$ $\quad R_0=3$

326. R: trapezoid $y=0,\ y=1,\ y=x,\ y=-x+3$; $\rho(x,y)=2x+y$

a) $I_x=\int_0^1\int_0^x y^2(2x+y)\,dy\,dx+\int_1^2\int_0^1 y^2(2x+y)\,dy\,dx+\int_2^3\int_0^{3-x} y^2(2x+y)\,dy\,dx$

$=\int_0^1\left(\frac{2}{3}xy^3+\frac{1}{4}y^4\right|_0^x dx+\int_1^2\left(\frac{2}{3}xy^3+\frac{1}{4}y^4\right|_0^1 dx+\int_2^3\left(\frac{2}{3}xy^3+\frac{1}{4}y^4\right|_0^{3-x}dx$

$=\int_0^1\frac{11}{12}x^4\,dx+\int_1^2\left(\frac{2}{3}x+\frac{1}{4}\right)dx+\int_2^3\left(-\frac{2}{3}x^4+6x^3-18x^2+18x\right)dx$

$=\left(\frac{11}{60}x^5\right|_0^1+\left(\frac{1}{3}x^2+\frac{1}{4}x\right|_1^2+\left(-\frac{2}{15}x^5+\frac{3}{2}x^4-6x^3+9x^2\right|_2^3$

$=\frac{11}{60}+\left(\frac{4}{3}+\frac{1}{2}\right)-\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{-486}{15}+\frac{243}{2}-162+81\right)-\left(\frac{-64}{15}+24-48+36\right)$

$=\frac{37}{20}$

$$I_y = \int_0^1\int_0^x x^2(2x+y)\,dy\,dx + \int_1^2\int_0^1 x^2(2x+y)\,dy\,dx + \int_2^3\int_0^{3-x} x^2(2x+y)\,dy\,dx$$

$$= \int_0^1 2x^3y + \tfrac{1}{2}x^2y^2\Big|_0^x dx + \int_1^2 (2x^3y + \tfrac{1}{2}x^2y^2\Big|_0^1 dx + \int_2^3 (2x^3y + \tfrac{1}{2}x^2y^2\Big|_0^{3-x} dx$$

$$= \int_0^1 \tfrac{5}{2}x^4\,dx + \int_1^2 (2x^3 + \tfrac{1}{2}x^2)\,dx + \int_2^3 (-\tfrac{3}{2}x^4 + 3x^3 + \tfrac{9}{2}x^2)\,dx$$

$$= (\tfrac{1}{2}x^5\Big|_0^1 + (\tfrac{1}{2}x^4 + \tfrac{1}{6}x^3\Big|_1^2 + (-\tfrac{3}{10}x^5 + \tfrac{3}{4}x^4 + \tfrac{3}{2}x^3\Big|_2^3$$

$$= \tfrac{1}{2} + (8 + \tfrac{4}{3}) - (\tfrac{1}{2} + \tfrac{1}{6}) + (-\tfrac{729}{10} + \tfrac{243}{4} + \tfrac{81}{2}) - (-\tfrac{48}{5} + 12 + 12)$$

$$= \frac{1387}{60}$$

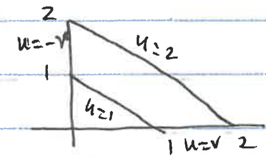
$$I_0 = I_x + I_y$$

$$= \tfrac{37}{20} + \tfrac{1387}{60}$$

$$= \tfrac{749}{30}$$

b) $R_x = \sqrt{\dfrac{37/20}{41/6}} = \dfrac{\sqrt{45510}}{410}$ $\quad R_y = \sqrt{\dfrac{\frac{1387}{60}}{41/6}} = \dfrac{\sqrt{568670}}{410}$ $\quad R_0 = \sqrt{\dfrac{749/30}{41/6}} = \dfrac{\sqrt{153545}}{205}$

397. $u = x+y,\ v = x-y$



$u = x+y \quad\Rightarrow\quad x = u-y \qquad$ so $\quad u-y = v+y \qquad\qquad x = u - \frac{u-v}{2}$

$v = x-y \qquad\quad\ x = v+y \qquad\qquad\quad 2y = u-v \qquad\qquad x = \frac{u+v}{2}$

$y = \frac{u-v}{2}$

$$|J(u,v)| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left| -\tfrac{1}{4} - \tfrac{1}{4} \right|$$

$$= \frac{1}{2}$$

so $\iint_R (x^3 + 3x^2y + 3xy^2 + y^3)\, dA$

$$= \int_1^2 \int_{-u}^{u} \frac{1}{2} u^3 \, dv\, du$$

$$= \int_1^2 \frac{1}{2} v u^3 \Big|_{-u}^{u} du$$

$$= \int_1^2 u^4 \, du$$

$$= \frac{1}{5} u^5 \Big|_1^2$$

$$= \frac{32}{5} - \frac{1}{5}$$

$$= 31/5$$