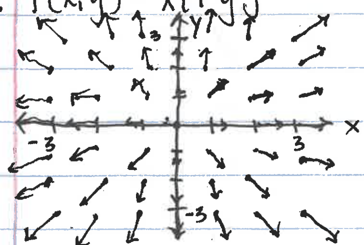
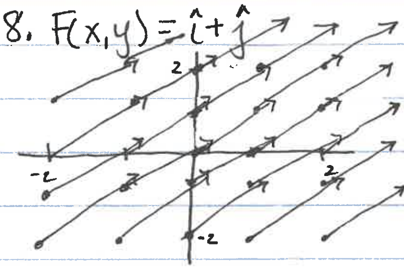
WP14 Solutions

5. $F(x,y) = x\hat{i} + y\hat{j}$



8. $F(x,y) = \hat{i} + \hat{j}$



23. All vectors point toward origin and have constant length

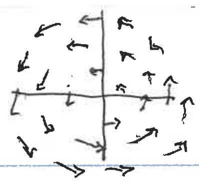
$$F(x,y) = \frac{y}{\sqrt{x^2+y^2}}\hat{i} - \frac{x}{\sqrt{x^2+y^2}}\hat{j}$$

25. $F(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

$F(0,0)=0$ & $F(a,b)$ is tangent to $x^2+y^2=a^2+b^2$, mag $= \sqrt{a^2+b^2}$

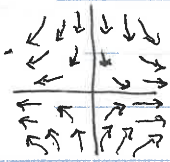
$$F(x,y) = y\hat{i} - x\hat{j}$$

32.



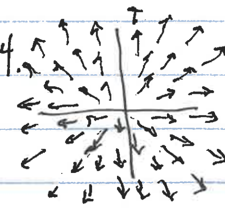
$G(x,y) = -y\hat{i} + x\hat{j}$

33.



$H(x,y) = x\hat{i} - y\hat{j}$

34.



$F(x,y) = x\hat{i} + y\hat{j}$

51. $F(x,y,z) = -\frac{1}{2}x\hat{i} - \frac{1}{2}y\hat{j} + \frac{1}{4}k$

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

from $(1,0,0)$ to $(-1,0,3\pi)$

$$\int_C F\cdot dr = \int_0^{3\pi} \langle -\tfrac{1}{2}\cos t, -\tfrac{1}{2}\sin t, \tfrac{1}{4}\rangle \cdot \langle -\sin t, \cos t, 1\rangle\, dt$$

$$= \int_0^{3\pi} \left(\tfrac{1}{2}\sin t\cos t - \tfrac{1}{2}\sin t\cos t + \tfrac{1}{4}\right) dt$$

$$= \int_0^{3\pi} \tfrac{1}{4}\, dt$$

$$= \tfrac{1}{4}t \Big|_0^{3\pi}$$

$$= \tfrac{3\pi}{4}$$

54. wire shaped as a circle of radius 2 centered at $(3,4)$ with $\rho(x,y) = y^2$

$r(t) = \langle 2\cos t + 3, 2\sin t + 4\rangle$

$$\int_0^{2\pi} (2\sin t+4)^2 \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt$$

$$= \int_0^{2\pi} (8\sin^2 t + 32\sin t + 32)\, dt$$

$$= \int_0^{2\pi} (4 - 4\cos 2t + 32\sin t + 32)\, dt$$

$$= (4t - 2\sin 2t - 32\cos t + 32t \Big|_0^{2\pi}$$

$$= (8\pi - 0 - 32 + 64\pi) - (0 - 0 - 32 + 0)$$

$$= 72\pi$$

57. $\int_C yz\,dx + xz\,dy + xy\,dz$
over $(1,1,1)$ to $(3,2,0)$

$r(t) = \langle 2t+1, t+1, -t-1 \rangle$
$r'(t) = \langle 2, 1, -1 \rangle$
$\int_0^1 [(t+1)(-t+1)2 + (2t+1)(-t+1)1 + (2t+1)(t+1)(-1)]dt$
$= \int_0^1 (-2t^2+2 - 2t^2+t+1 - 2t^2-3t-1)dt$
$= \int_0^1 (-6t^2-2t+2)\,dt$
$= (-2t^3 - t^2 + 2t \Big|_0^1$
$= (-2-1+2) - 0$
$= -1$

94. wire bent into semicircle of radius a; linear mass density at point P is proportional to distance from line through endpoints. Find mass

$r(t) = \langle a\cos t, a\sin t \rangle$ $\quad \rho(x,y) = K\sqrt{x^2+y^2}$ $\quad$ for $K>0$
$r'(t) = \langle -a\sin t, a\cos t \rangle$
$\int_0^{2\pi} K\sqrt{(a\cos t)^2 + (a\sin t)^2}\,\sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt$

$= \int_0^{2\pi} K\sqrt{a^2(\cos^2 t + \sin^2 t)}\,\sqrt{a^2(\sin^2 t + \cos^2 t)}\,dt$

$= \int_0^{2\pi} Ka^2\,dt$
$= Ka^2 t \Big|_0^{2\pi}$
$= 2\pi Ka^2$

GP15 Solutions

102. $F(x,y,z) = y\hat{i} + (x+z)\hat{j} - y\hat{k}$ is conservative?

No. consider the paths
$r_1(t) = \langle t, t, t \rangle$ and
$r_2(t) = \langle t, t^2, t \rangle$

$$\int_{C_1} F \cdot dr = \int_0^1 \langle t, 2t, -t \rangle \cdot \langle 1,1,1 \rangle dt = \int_0^1 2t\, dt = t^2 \Big|_0^1 = 1$$

$$\int_{C_2} F \cdot dr = \int_0^1 \langle t^2, 2t, -t^2 \rangle \langle 1, 2t, 1 \rangle dt = \int_0^1 4t^2\, dt = \frac{4}{3}t^3 \Big|_0^1 = 4/3$$

so $\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$

113. $\oint_C (2y\,dx + 2x\,dy)$ C: line from $(0,0)$ to $(4,4)$

$f(x,y) = 2xy$
$r(t) = \langle t, t \rangle$ for $t \in [0,4]$
so $\int_C \nabla f \cdot dr$
$$= f(r(4)) - f(r(0)) = f(4,4) - f(0,0) = 32 - 0 = 32$$

116. $F(x,y) = (12xy)\hat{i} + 6(x^2+y^2)\hat{j}$

test $r_1(t) = \langle t, t \rangle$ and $r_2(t) = \langle t, t^2 \rangle$ on $(0,1)$

$$\int_{C_1} F \cdot dr = \int_0^1 \langle 12t^2, 12t^2 \rangle \langle 1,1 \rangle dt = \int_0^1 24t^2\, dt = 8t^3 \Big|_0^1 = 8$$

$$\int_{C_2} F \cdot dr = \int_0^1 \langle 12t^3, 6(t^2+t^4) \rangle \langle 1, 2t \rangle dt = \int_0^1 24t^3 + 12t^5\, dt = 6t^4 + 2t^6 \Big|_0^1 = 8$$

Conservative field

$$\int 12xy\, dx = 6x^2y + g(y)$$
$$6x^2 + g_y = 6(x^2+y^2)$$
$$g_y = 6y^2$$
$$g(y) = 2y^3$$

so $f(x,y) = 6x^2y + 2y^3$ is the potential function

119. $F(x,y,z) = (ye^z)\hat{i} + (xe^z)\hat{j} + (xye^z)\hat{k}$

test $r_1(t) = \langle t,t,t\rangle$ and $r_2(t) = \langle t,t,t^2\rangle$ for $t\in[0,1]$

$\int_{C_1} F\cdot dr = \int_0^1 \langle te^t, te^t, t^2e^t\rangle\langle 1,1,1\rangle dt$
$= \int_0^1 (2t+t^2)e^t dt$ (use IBP)
$= (2t+t^2)e^t - (2+2t)e^t + 2e^t \Big|_0^1$
$= (3e-4e+2e) - (0-2+2)$
$= e$

$\int_{C_2} F\cdot dr = \int_0^1 \langle te^{t^2}, te^{t^2}, t^2e^{t^2}\rangle\langle 1,1,2t\rangle dt$
$= \int_0^1 (2t+2t^3)e^{t^2} dt$ (use IBP & Sub)
$= (1+t^2)e^{t^2} - e^{t^2}\Big|_0^1$
$= (2e-e) - (1-1)$
$= e$

so this is a conservative vector field

to get potential $\int ye^z dx = xye^z + g(y,z)$

need partial wrt y: $xe^z + g_y = xe^z$ so $g_y = 0$

thus $g(y,z) = h(z)$

$f_z = xye^z + h'z \Rightarrow h'(z) = 0$

So $f(x,y,z) = xye^z$ is the potential function

126. $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ C: path from $(1,\frac{1}{2},2)$ to $(2,1,-1)$

So $r(t_i) = (1,\frac{1}{2},2)$ and $r(t_f) = (2,1,-1)$

so $\int_C \nabla f\cdot dr = f(r(t_f)) - f(r(t_o))$
$= f(2,1,-1) - f(1,\frac{1}{2},2)$
$= [\cos(2\pi) + \sin\pi - 2(1)(-1)] - [\cos(\pi) + \sin(\frac{\pi}{2}) - 1(\frac{1}{2})(2)]$
$= (1+0+2) - (-1+1-1)$
$= 4$

143. $F = -y\hat{i} + x\hat{j}$ around and across $r_1(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}$; $0\le t\le\pi$ and $r_2(t) = t\hat{i}$ $-a\le t\le a$

Flux $= \int_C F\cdot N ds$
$= \int_0^\pi \langle -a\sin t, a\cos t\rangle\langle a\cos t, a\sin t\rangle dt + \int_{-a}^a \langle 0,t\rangle\langle 1,0\rangle dt$
$= \int_0^\pi (-a^2\sin t\cos t + a^2\sin t\cos t) dt + \int_{-a}^a 0\, dt$
$= \int_0^\pi 0\, dt + 0t\Big|_{-a}^a$
$= 0$

Flux $= 0$

143 continued

Circulation $= \int_C F \cdot T\,ds$

$= \int^{\pi} \langle -a\sin t, a\cos t\rangle \cdot \langle -a\sin t, a\cos t\rangle\,dt + \int_{-a}^{a} \langle 0, t\rangle \cdot \langle 1, 0\rangle\,dt$

$= \int_0^{\pi} (a^2\sin^2 t + a^2\cos^2 t)\,dt + \int_{-a}^{a} 0 + 0\,dt$

$= \int_0^{\pi} a^2\,dt \quad + 0$

$= a^2 t \Big|_0^{\pi}$

Circulation $= \pi a^2$