WP16 Solutions

216. $F(x,y,z) = (x\cos y)\hat{i} + (xy^2)\hat{j}$

$\text{curl } F = (0-0)\hat{i} + (0-0)\hat{j} + (y^2 + x\sin y)\hat{k}$
$= (y^2 + x\sin y)\hat{k}$

218. $F(x,y,z) = xyz\hat{i} + x^2y^2z^2\hat{j} + y^2z^3\hat{k}$

$\text{Curl } F = (2yz^3 - 3y^2z^2)\hat{i} + (xy-0)\hat{j} + (2xy^2z^2 - xz)\hat{k}$
$= (2yz^3 - 3y^2z^2)\hat{i} + xy\hat{j} + (2xy^2z^2 - xz)\hat{k}$

223. $F(x,y,z) = 3xyz^2\hat{i} + y^2\sin z\hat{j} + xe^{2z}\hat{k}$

$\text{div } F = 3yz^2 + 2y\sin z + 2xe^{z}$

230. $F(x,y,z) = xyz\hat{i} + x^2y^2z^2\hat{j} + y^2z^3\hat{k}$

$\text{div } F = yz + 2x^2yz^2 + 3y^2z^2$

234. $F(x,y,z) = 2\hat{i} + 2x\hat{j} + 3y\hat{k}$
$G(x,y,z) = x\hat{i} - y\hat{j} + z\hat{k}$

$$F \times G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2x & 3y \\ x & -y & z \end{vmatrix}$$

$= (2xz + 3y^2)\hat{i} - (2z - 3xy)\hat{j} + (-2y - 2x^2)\hat{k}$

$\text{curl}(F \times G)$
$= (-2-2)\hat{i} + (2x - 4x)\hat{j} + (3y - 6y)\hat{k}$
$= -4\hat{i} - 2x\hat{j} - 3y\hat{k}$

235. $F(x,y,z) = 2\hat{i} + 2x\hat{j} + 3y\hat{k}$
$G(x,y,z) = x\hat{i} - y\hat{j} + z\hat{k}$

$$F \times G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2x & 3y \\ x & -y & z \end{vmatrix}$$

$= (2xz + 3y^2)\hat{i} - (2z - 3xy)\hat{j} + (-2y - 2x^2)\hat{k}$

$\text{div}(F \times G)$
$= 2z + 3x + 0$
$= 2z + 3x$

255. $F(x,y,z) = (3x^2y + az)\hat{i} + x^3\hat{j} + (3x + 3z^2)\hat{k}$
Find a so that F is conservative

$\text{Curl } F = (0-0)\hat{i} + (a-3)\hat{j} + (3x^2 - 3x^2)\hat{k}$
$= (a-3)\hat{j}$

need $\text{curl } F = 0$ for F to be conservative
so $a = 3$

275. Plane: $2x-4y+3z=16$

let $u=2x$ and $v=4y$

then $u-v+3z=16$

$3z=16-u+v$

$z=\frac{1}{3}(16-u+v)$

so $r(u,v)=\langle u, v, \frac{1}{3}(16-u+v)\rangle$

for $-\infty<u,v<\infty$

277. portion of cylinder $x^2+y^2=9$ in first octant, $0\le z\le 3$

let $3\cos u=x$, $3\sin u=y$, and $v=z$

then $x^2+y^2=9$

so $r(u,v)=\langle 3\cos u, 3\sin u, v\rangle$

$0\le u\le \pi/2$, $0\le v\le 3$

281. $S: x^2+y^2+z^2=4$, $z\ge 0$

$\iint_S z\, dS$

$r(u,v)=\langle 2\cos v\sin u, 2\sin v\sin u, 2\cos u\rangle$ $\quad 0\le u\le \pi/2$, $0\le v\le 2\pi$

$t_u=\langle 2\cos v\cos u, 2\sin v\cos u, -2\sin u\rangle$ $\quad t_v=\langle -2\sin v\sin u, 2\cos v\sin u, 0\rangle$

so $t_u\times t_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos v\cos u & 2\sin v\cos u & -2\sin u \\ -2\sin v\sin u & 2\cos v\sin u & 0 \end{vmatrix}$

$= 4\cos v\sin^2 u\,\hat{i} + 4\sin v\sin^2 u\,\hat{j} + (4\cos^2 v\sin u\cos u + 4\sin^2 v\sin u\cos u)\hat{k}$

$= 4\cos v\sin^2 u\,\hat{i} + 4\sin v\sin^2 u\,\hat{j} + 4\sin u\cos u\,\hat{k}$

$\|t_u\times t_v\| = \sqrt{16\cos^2 v\sin^4 u + 16\sin^2 v\sin^4 u + 16\sin^2 u\cos^2 u}$

$= 4\sin u$

$\iint_S z\,dS = \int_0^{\pi/2}\int_0^{2\pi} 2\cos u(4\sin u)\,dv\,du$

$=\int_0^{\pi/2} 8v\cos u\sin u \Big|_0^{2\pi} du$

$=16\pi\int_0^{\pi/2}\cos u\sin u\,du$

$=16\pi\left(\frac{1}{2}\sin^2 u\Big|_0^{\pi/2}\right.$

$=8\pi(1-0)$

$=8\pi$

284. $F(x,y,z) = x\hat{i} + 2y\hat{j} - 3z\hat{k}$; $15x - 12y + 3z = 6$ above $0 \le x \le 1, 0 \le y \le 1$

Let $P(x,y,z) = x$, $Q(x,y,z) = 2y$, $R(x,y,z) = -3z$

$z = 2 - 5x + 4y$ so $z_x = -5$ and $z_y = 4$

then $\iint_S F \cdot N\,dS = \int_0^1\int_0^1 [-x(-5) - 2y(4) - 3(2-5x+4y)]\,dy\,dx$

$= \int_0^1\int_0^1 (20x - 20y - 6)\,dy\,dx$

$= \int_0^1 (20xy - 10y^2 - 6y\Big|_0^1\,dx$

$= \int_0^1 (20x - 16)\,dx$

$= (10x^2 - 16x\Big|_0^1$

$= 10 - 16$

$= -6$

285. $F(x,y,z) = x\hat{i} + y\hat{j}$, $z = \sqrt{1-x^2-y^2}$ (use polar coordinates)

Let $P(x,y,z) = x \Rightarrow r\cos\theta$, $Q(x,y,z) = y \Rightarrow r\sin\theta$, $R(x,y,z) = 0$

$z = \sqrt{1-x^2-y^2} \Rightarrow \sqrt{1-r^2}$

$z_x = \dfrac{x}{\sqrt{1-x^2-y^2}} \Rightarrow \dfrac{r\cos\theta}{\sqrt{1-r^2}}$ $\quad z_y = \dfrac{y}{\sqrt{1-x^2-y^2}} \Rightarrow \dfrac{r\sin\theta}{\sqrt{1-r^2}}$

then $\iint_S F \cdot N\,dS = \int_0^{2\pi}\int_0^1 \left(-\dfrac{r^2\cos^2\theta}{\sqrt{1-r^2}} - \dfrac{r^2\sin^2\theta}{\sqrt{1-r^2}} + 0\right) r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 -r^3(1-r^2)^{-1/2}\,dr\,d\theta$ $\quad$ let $u = 1-r^2$

$du = -2r\,dr$

$= \int_0^{2\pi}\int_0^1 \left(-\frac{1}{2}u^{1/2} + \frac{1}{2}u^{-1/2}\right) du\,d\theta$

$= \int_0^{2\pi} \left(-\frac{1}{3}u^{3/2} + u^{1/2}\Big|_0^1\,d\theta$

$= \int_0^{2\pi} \left(-\frac{1}{3} + 1\right) d\theta$

$= \frac{2}{3}\theta\Big|_0^{2\pi}$

$= \frac{4\pi}{3}$

303. $\iint_S F\cdot N\,dS$; $F(x,y,z)=xy\hat{i}+z\hat{j}+(x+y)\hat{k}$; S: $x+y+z=1$ in first octant.

Let $P(x,y,z)=xy$, $Q(x,y,z)=z$, $R(x,y,z)=x+y$

$r(u,v)=\langle uv, 1-u-v, u+v\rangle$ $\quad r_u=\langle 1,0,-1\rangle$ and $r_v=\langle 0,1,-1\rangle$

$$r_u\times r_v=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&0&-1\\0&1&-1\end{vmatrix}$$

$0\le u\le 1$ and $0\le v\le 1-u$

$$=\langle 1,1,1\rangle$$

so $\iint_S F\cdot N\,dS=\int_0^1\int_0^{1-u}\langle uv, 1-u-v, u+v\rangle\langle 1,1,1\rangle\,dv\,du$

$$=\int_0^1\int_0^{1-u}(1+uv)\,dv\,du$$
$$=\int_0^1\left(v+\tfrac{1}{2}uv^2\right|_0^{1-u}du$$
$$=\int_0^1\left[(1-u)+\tfrac{1}{2}u(1-u)^2\right]du$$
$$=\int_0^1\left(\tfrac{1}{2}u^3-u^2-\tfrac{1}{2}u+1\right)du$$
$$=\tfrac{1}{8}u^4-\tfrac{1}{3}u^3-\tfrac{1}{4}u^2+u\Big|_0^1$$
$$=\left(\tfrac{1}{8}-\tfrac{1}{3}-\tfrac{1}{4}+1\right)-0$$
$$=\frac{13}{24}$$

313. $\iint_S x^2yz\,dS$; S is part of plane $z=1+2x+3y$ above $0\le x\le 3, 0\le y\le 2$

so $x=x$, $y=y$, $z=1+2x+3y$

$\frac{\partial z}{\partial x}=2$ and $\frac{\partial z}{\partial y}=3$

$$\iint_S x^2yz\,dS=\int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{1+4+9}\,dy\,dx$$
$$=\sqrt{14}\int_0^3\int_0^2(x^2y+2x^3y+3x^2y^2)\,dy\,dx$$
$$=\sqrt{14}\int_0^3\left(\tfrac{1}{2}x^2y^2+x^3y^2+x^2y^3\right|_0^2dx$$
$$=\sqrt{14}\int_0^3(2x^2+4x^3+8x^2)\,dx$$
$$=\sqrt{14}\int_0^3(10x^2+4x^3)\,dx$$
$$=\sqrt{14}\left(\tfrac{10}{3}x^3+x^4\right|_0^3$$
$$=\sqrt{14}(90+81)-\sqrt{14}(0)$$
$$=171\sqrt{14}$$

315. $\iint_S yz\,dS$; S is part of plane $z=y+3$ that lies inside cylinder $x^2+y^2=1$.

Let $x=x,\ y=y,\ z=y+3$ $\qquad \frac{\partial z}{\partial x}=0,\ \frac{\partial z}{\partial y}=1$

so $\iint_S yz\,dS = \iint_R y(y+3)\sqrt{1+0^2+1^2}\,dA$ $\qquad$ change to polar

$$= \int_0^{2\pi}\int_0^1 r\sin\theta(r\sin\theta+3)\sqrt{2}\,r\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (3r^2\sin\theta + r^3\sin^2\theta)\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\left(r^3\sin\theta + \tfrac{1}{4}r^4\sin^2\theta \right|_0^1 d\theta$$
$$= \sqrt{2}\int_0^{2\pi}(\sin\theta + 1/4\sin^2\theta)\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}(\sin\theta + \tfrac{1}{8} - \tfrac{1}{8}\cos 2\theta)\,d\theta$$
$$= \sqrt{2}\left(-\cos\theta + \tfrac{1}{8}\theta - \tfrac{1}{16}\sin 2\theta \right|_0^{2\pi}$$
$$= \sqrt{2}(-1+\pi/4-0) - \sqrt{2}(-1+0-0)$$
$$= \frac{\sqrt{2}\pi}{4}$$

317. $\iint_S (x\hat{i}+y\hat{j})\cdot dS$; S is the surface $x^2+y^2=4$, $1\le z\le 3$ w/outward normals

Let $x=2\cos u,\ y=2\sin u,\ z=v$ $\qquad u\in[0,2\pi]\quad v\in[1,3]$

so $r_u=\langle -2\sin u, 2\cos u, 0\rangle$ and $r_v=\langle 0,0,1\rangle$

the outward orientation gives

$$r_u\times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=\langle 2\cos u, 2\sin u, 0\rangle$$

$$\iint_S (x\hat{i}+y\hat{j})\cdot dS = \int_0^{2\pi}\int_1^3 \langle 2\cos u, 2\sin u, 0\rangle\cdot\langle 2\cos u, 2\sin u, 0\rangle\,dv\,du$$
$$= \int_0^{2\pi}\int_1^3 (4\cos^2 u + 4\sin^2 u + 0)\,dv\,du$$
$$= \int_0^{2\pi}\int_1^3 4\,dv\,du$$
$$= \int_0^{2\pi} 4v\Big|_1^3\,du$$
$$= \int_0^{2\pi}(12-4)\,du$$
$$= 8u\Big|_0^{2\pi}$$
$$= 16\pi$$