WP 17 Solutions

327 $F(x,y,z) = \langle z, x, y \rangle$; S is hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$

$r(t) = \langle a\cos t, a\sin t, 0 \rangle \quad 0 \le t \le 2\pi$

then $r'(t) = \langle -a\sin t, a\cos t, 0 \rangle$

$F(r(t)) = \langle 0, a\cos t, a\sin t \rangle$

so

$$\iint_S \text{curl } F \cdot dS = \int_0^{2\pi} \langle 0, a\cos t, a\sin t \rangle \cdot \langle -a\sin t, a\cos t, 0 \rangle \, dt$$
$$= \int_0^{2\pi} (0 + a^2\cos^2 t + 0)\, dt$$
$$= \int_0^{2\pi} a^2 \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right) dt$$
$$= a^2\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t \right|_0^{2\pi}$$
$$= a^2(\pi + 0) - a^2(0 - 0)$$
$$= \pi a^2$$

331. $F(x,y,z) = \langle 2y, -6z, 3x \rangle$; S is paraboloid $z = 4 - x^2 - y^2$ above xy-plane

let $r(t) = \langle 2\cos t, 2\sin t, 0 \rangle$ for $0 \le t \le 2\pi$

then $r'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$F(r(t)) = \langle 4\sin t, 0, 6\cos t \rangle$

So

$$\iint_S \text{curl } F \cdot dS = \int_0^{2\pi} \langle 4\sin t, 0, 6\cos t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$
$$= \int_0^{2\pi} (-8\sin^2 t + 0 + 0)\, dt$$
$$= \int_0^{2\pi} -8\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t\right) dt$$
$$= -4t + 2\sin 2t \Big|_0^{2\pi}$$
$$= (-8\pi + 0) - (0 + 0)$$
$$= -8\pi$$

333. $F(x,y,z) = xy\hat{i} + x^2\hat{j} + z^2\hat{k}$; C is intersection of $z = x^2+y^2$ and $z=y$

let $r(t) = \langle \cos t, \sin t, 1\rangle$ for $0 \le t \le 2\pi$

then $r'(t) = \langle -\sin t, \cos t, 0\rangle$

$F(r(t)) = \langle \cos t \sin t, \cos^2 t, 1\rangle$

so

$$\iint_S \text{curl } F \cdot dS = \int_0^{2\pi} \langle \cos t\sin t, \cos^2 t, 1\rangle \cdot \langle -\sin t, \cos t, 0\rangle \, dt$$
$$= \int_0^{2\pi} (-\cos t \sin^2 t + \cos^3 t + 0)\, dt$$
$$= \int_0^{2\pi} [-\cos t \sin^2 t + \cos t - \cos t \sin^2 t]\, dt$$
$$= \int_0^{2\pi} (\cos t - 2\cos t \sin^2 t)\, dt$$
$$= (\sin t - \tfrac{2}{3}\sin^3 t \Big|_0^{2\pi}$$
$$= (0-0) - (0-0)$$
$$= 0$$

334. $F(x,y,z) = 4y\hat{i} + z\hat{j} + 2y\hat{k}$ C is the intersection of $x^2+y^2+z^2=4$ and $z=0$

let $r(t) = \langle 2\cos t, 2\sin t, 0\rangle$ $0 \le t \le 2\pi$

then $r'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$F(r(t)) = \langle 8\sin t, 0, 4\sin t\rangle$

So

$$\iint_S \text{curl } F \cdot dS = \int_0^{2\pi} \langle 8\sin t, 0, 4\sin t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle$$
$$= \int_0^{2\pi} (-16\sin^2 t + 0 + 0)\, dt$$
$$= \int_0^{2\pi} -16(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t)\, dt$$
$$= (-8t + 4\sin 2t \Big|_0^{2\pi}$$
$$= (-16\pi + 0) - (0+0)$$
$$= -16\pi$$

353. $F(x,y,z) = y\hat{i} + z\hat{j} + x\hat{k}$; C is the triangle $(0,0,0),(2,0,0)$ & $(0,-2,2)$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$= \langle -1, -1, -1 \rangle$

S is the portion of the plane $y+z=0$

so, $0 \le y \le 2$ and $-y \le z \le 0$

by Stokes' Thm

$$\iint_S \text{curl} F \cdot dS = \iint_R [-(-1)(0) - (-1)(-1) - 1] \, dA$$

$$= \int_0^2 \int_{-y}^0 -2 \, dz \, dy$$

$$= \int_0^2 -2z \Big|_{-y}^0 dy$$

$$= \int_0^2 -2y \, dy$$

$$= -y^2 \Big|_0^2$$

$$= -4$$

356. $F(x,y,z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$; S is part of plane $x+y+z=1$ in positive octant

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$= \langle 0, 0, 1-2y \rangle$

$z = 1-x-y$

so $0 \le x \le 1$ and $0 \le y \le 1-x$

by Stokes' Thm

$$\iint_S \text{curl} F \cdot dS = \iint_R [-(0)(-1) - (0)(-1) + (1-2y)] \, dA$$

$$= \int_0^1 \int_0^{1-x} (1-2y) \, dy \, dx$$

$$= \int_0^1 y - y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 [1 - x - (1 - 2x + x^2)] \, dx$$

$$= \int_0^1 (x - x^2) \, dx$$

$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right) \Big|_0^1$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0-0)$$

$$= \tfrac{1}{6}$$

386. $F(x,y,z) = \langle x^4, -x^3z^2, 4xy^2z \rangle$ $x^2+y^2=1$, $z=x+2$, $z=0$

$\text{div} F = 4x^3 + 0 + 4xy^2$
$= 4x(x^2+y^2)$

using cylindrical coords

$0 \le \theta \le 2\pi$, $0 \le r \le 1$, $0 \le z \le 2 + r\cos\theta$

$$\iiint_E \text{div} F \, dV = \int_0^{2\pi}\int_0^1\int_0^{2+r\cos\theta} 4r\cos\theta(r^2)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 4r^4\cos\theta\, z \Big|_0^{2+r\cos\theta} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (8r^4\cos\theta + 4r^5\cos^2\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{8}{5}r^5\cos\theta + \frac{2}{3}r^6\cos^2\theta \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{8}{5}\cos\theta + \frac{1}{3} + \frac{1}{3}\cos 2\theta\right) d\theta$$

$$= \frac{8}{5}\sin\theta + \frac{1}{3}\theta + \frac{1}{6}\sin 2\theta \Big|_0^{2\pi}$$

$$= (0 + 2\pi/3 + 0) - (0+0+0)$$

$$= 2\pi/3$$

391. $F(x,y,z) = \langle 0, y, -z \rangle$ $y = x^2+z^2$, $0 \le y \le 1$; $x^2+z^2 \le 1$, $y=1$

$\text{div} F = 0 - 1 - 1 = -2$ (note: negative due to normal)

in cylindrical: $0 \le \theta \le 2\pi$, $0 \le r \le 1$, $1-r^2 \le z \le 1$

$$\iiint_E \text{div} F \, dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^1 -2r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -2r(1-(1-r^2))\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -2r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}r^4 \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\, d\theta$$

$$= -\frac{1}{2}(2\pi)$$

$$= -\pi$$

392. $F(x,y,z) = \langle 0, x+y, z^4 \rangle$ cone: $z = \sqrt{x^2+y^2}$ beneath $z=1$

$\operatorname{div} F = 0 + 1 - 4\cancel{z}^3$
$= 1 - 4z^3$

region: $r \le z \le 1$, $0 \le r \le 1$
$0 \le \theta \le 2\pi$

(note negative orientation)

$$\iiint_E \operatorname{div} F \, dV = \int_0^{2\pi}\int_0^1\int_r^1 (1-4z^3)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (z - z^4)\, r \Big|_r^1 \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^5 - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}r^{6} - \frac{1}{3}r^3 \Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{6}\, d\theta$$

$$= -\frac{1}{6}(2\pi)$$

$$= -\pi/3$$

403. $F(x,y,z) = \langle (x^3 - 3y), 2yz+1, xyz \rangle$ cube; $x = \pm 1$, $y = \pm 1$, $z = \pm 1$

$\operatorname{div} F = 3x^2 + 2z + xy$

$$\iiint_E \operatorname{div} F\, dV = \int_{-1}^1\int_{-1}^1\int_{-1}^1 (3x^2 + 2z + xy)\, dz\, dy\, dx$$

$$= \int_{-1}^1\int_{-1}^1 (3x^2 z + z^2 + xyz \Big|_{-1}^1 \, dy\, dx$$

$$= \int_{-1}^1\int_{-1}^1 (6x^2 + 2xy)\, dy\, dx$$

$$= \int_{-1}^1 6x^2 y + xy^2 \Big|_{-1}^1 \, dx$$

$$= \int_{-1}^1 12x^2\, dx$$

$$= 4x^3 \Big|_{-1}^1$$

$$= 4 - (-4)$$

$$= 8$$

412. $F(x,y,z) = \langle x^3, y^3, z^3 \rangle$ through sphere $x^2+y^2+z^2=1$

$\text{div} F = 3x^2+3y^2+3z^2$
$= 3(x^2+y^2+z^2)$

using spherical
$0 \le \theta \le 2\pi, \; 0 \le \phi \le \pi \quad 0 \le \rho \le 1$

$$\iiint_E \text{div} F \, dV = \int_0^1 \int_0^\pi \int_0^{2\pi} 3\rho^2 \cdot \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

$$= \int_0^1 \int_0^\pi 3\rho^4 \theta \sin\phi \Big|_0^{2\pi} d\phi \, d\rho$$

$$= \int_0^1 \int_0^\pi 6\pi\rho^4 \sin\phi \, d\phi \, d\rho$$

$$= \int_0^1 -6\pi\rho^4 \cos\phi \Big|_0^\pi d\rho$$

$$= \int_0^1 -6\pi\rho^4(-1) - (-6\pi\rho^4) \, d\rho$$

$$= \int_0^1 12\pi\rho^4 \, d\rho$$

$$= \frac{12\pi}{5}\rho^5 \Big|_0^1$$

$$= \frac{12\pi}{5}$$

415. $F(x,y,z) = \langle 3z^2, 6, 6xz \rangle$ on $S: y = x^2, \; 0 \le x \le 2, \; 0 \le z \le 3$

$\text{div} F = 0+0+6x = 6x$

$$\iiint_E \text{div} F \, dV = \int_0^3 \int_0^2 \int_0^{x^2} 6x \, dy \, dx \, dz$$

$$= \int_0^3 \int_0^2 6xy \Big|_0^{x^2} dx \, dz$$

$$= \int_0^3 \int_0^2 6x^3 \, dx \, dz$$

$$= \int_0^3 \frac{3}{2}x^4 \Big|_0^2 dz$$

$$= \int_0^3 24 \, dz$$

$$= 24z \Big|_0^3$$

$$= 72.$$