WP4 Solutions

103. $r(t) = t^2\hat{i} + (2t^2+1)\hat{j}$ for $1 \le t \le 3$

$r'(t) = 2t\hat{i} + 4t\hat{j}$

$s(t) = \int_1^3 \sqrt{(2t)^2+(4t)^2}\,dt$

$= \int_1^3 t\sqrt{20}\,dt$

$= \frac{1}{2}t^2\sqrt{20}\,\Big|_1^3$

$= \frac{9}{2}\sqrt{20} - \frac{1}{2}\sqrt{20}$

$= 8\sqrt{5}$

105. $r(t) = \langle t^2+1, 4t^3+3 \rangle$ for $-1 \le t \le 0$

$r'(t) = \langle 2t, 12t^2 \rangle$

$S(t) = \int_{-1}^0 \sqrt{4t^2 + 144t^4}\,dt$

$= \int_{-1}^0 2t\sqrt{1+36t^2}\,dt$

let $u = 1+36t^2$ so $du = 72t\,dt$

$t=0 \rightarrow u=1$ and $t=-1 \rightarrow u=37$

$= \frac{1}{36}\int_{37}^1 u^{1/2}\,du$

$= \frac{1}{54}u^{3/2}\Big|_{37}^1$

$= \frac{1}{54}(1-37^{3/2})$

107. $r(t) = \frac{1}{2}\cos t\,\hat{i} + \frac{1}{2}\sin t\,\hat{j} + \sqrt{\frac{3}{4}}\,t\,\hat{k}$ on $[0, 2\pi]$

$r'(t) = -\frac{1}{2}\sin t\,\hat{i} + \frac{1}{2}\cos t\,\hat{j} + \sqrt{3/4}\,\hat{k}$

$s(t) = \int_0^{2\pi} \sqrt{\frac{1}{4}\sin^2 t + \frac{1}{4}\cos^2 t + 3/4}\,dt$

$= \int_0^{2\pi} dt$

$= t\Big|_0^{2\pi}$

$= 2\pi$

109. $r(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 0\hat{k}$

$r'(t) = -3\sin t\,\hat{i} + 3\cos t\,\hat{j}$

$s(t) = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\,dt$

$= \int_0^{2\pi} 3\,dt$

$= 3t\Big|_0^{2\pi}$

$= 6\pi$

112. $r(t) = \langle 2\sin t, 5t, 2\cos t \rangle$ for $t \in [-10, 10]$

$r'(t) = \langle 2\cos t, 5, -2\sin t \rangle$

$s(t) = \int_{-10}^{10} \sqrt{4\cos^2 t + 25 + 4\sin^2 t}\,dt$

$= \int_{-10}^{10} \sqrt{29}\,dt$

$= \sqrt{29}\,t\Big|_{-10}^{10}$

$= \sqrt{29}(10) - \sqrt{29}(-10)$

$= 20\sqrt{29}$

115. $r(t) = \langle 2e^t, e^t\cos t, e^t\sin t \rangle$

$r'(t) = \langle 2e^t, e^t(\cos t - \sin t), e^t(\sin t + \cos t) \rangle$

$\|r'(t)\|$

$= \sqrt{4e^{2t} + e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2}$

$= \sqrt{4e^{2t} + e^{2t} + e^{2t}}$

$= e^t\sqrt{6}$

$T(t) = \dfrac{\langle 2e^t, e^t(\cos t - \sin t), e^t(\sin t + \cos t) \rangle}{e^t\sqrt{6}}$

$= \left\langle \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}(\cos t - \sin t), \frac{\sqrt{6}}{6}(\sin t + \cos t) \right\rangle$

117. $r(t) = \langle 2e^t, e^t\cos t, e^t\sin t \rangle$

From problem 115:

$T(t) = \left\langle \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}(\cos t - \sin t), \frac{\sqrt{6}}{6}(\sin t + \cos t) \right\rangle$

$T'(t) = \left\langle 0, -\frac{\sqrt{6}}{6}(\sin t + \cos t), \frac{\sqrt{6}}{6}(\cos t - \sin t) \right\rangle$

$\|T'(t)\| = \sqrt{\frac{1}{6}(\sin t + \cos t)^2 + \frac{1}{6}(\cos t - \sin t)^2}$

$= \sqrt{\frac{1}{6} + \frac{1}{6}}$

$= \sqrt{3}/3$

$N(t) = \left\langle 0, -\frac{\sqrt{2}}{2}(\sin t + \cos t), \frac{\sqrt{2}}{2}(\cos t - \sin t) \right\rangle$

$N(0) = \langle 0, -\sqrt{2}/2, \sqrt{2}/2 \rangle$

125. $r(t) = \langle 2\sin t, 5t, 2\cos t \rangle$

$r'(t) = \langle 2\cos t, 5, -2\sin t \rangle$

$\|r'(t)\| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t}$

$= \sqrt{29}$

$T(t) = \left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\sin t \right\rangle$

$T'(t) = \left\langle \frac{-2}{\sqrt{29}}\sin t, 0, -\frac{2}{\sqrt{29}}\cos t \right\rangle$

$\|T'(t)\| = \sqrt{4/29 \sin^2 t + 0 + 4/29 \cos^2 t}$

$= 2/29$

$N(t) = \langle -\sin t, 0, -\cos t \rangle$

133. $y = x - \frac{1}{4}x^2$ at $x = 2$

$y' = 1 - \frac{1}{2}x$

$y'' = -1/2$

$K = \frac{|-1/2|}{[1 + (1 - \frac{1}{2}x)^2]^{3/2}}$

$K = \frac{1/2}{(1 + 1 - x + \frac{1}{4}x^2)^{3/2}}$

$K = \frac{1}{2(2 - x + \frac{1}{4}x^2)^{3/2}}$

$K\big|_{x=2} = \frac{1}{2(2 - 2 + \frac{1}{4}(4))^{3/2}}$

$= \frac{1}{2(1)^{3/2}}$

$= \frac{1}{2}$

135. $r(t) = t\hat{i} + 6t^2\hat{j} + 4t\hat{k}$

$r'(t) = \hat{i} + 12t\hat{j} + 4\hat{k}$

$r''(t) = 12\hat{j}$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 12t & 4 \\ 0 & 12 & 0 \end{vmatrix}$$

$= -48\hat{i} + 0\hat{j} + 12\hat{k}$

$\|r' \times r''\| = \sqrt{2304 + 0 + 144}$

$= 12\sqrt{17}$

$\|r'\| = \sqrt{1 + 144t^2 + 16}$

$= \sqrt{17 + 144t^2}$

$$K = \frac{12\sqrt{17}}{(17 + 144t^2)^{3/2}}$$

139. $y = e^x$; limit as $x \to \infty$

$y = e^x$ so $y' = e^x$ and $y'' = e^x$

$$K = \frac{|e^x|}{[1 + (e^x)^2]^{3/2}}$$

$$= \frac{e^x}{(1 + e^{2x})^{3/2}}$$

$$\lim_{x \to \infty} K = \lim_{x \to \infty} \frac{e^x}{(1 + e^{2x})^{3/2}}$$

$= 0$

147. $y = \ln(x+1)$ @ $(2, \ln 3)$

$y' = \frac{1}{x+1}$ and $y'' = \frac{-1}{(x+1)^2}$

$$K = \frac{\left|\frac{-1}{(x+1)^2}\right|}{\left[1 + \left(\frac{1}{x+1}\right)^2\right]^{3/2}}$$

$$= \frac{1}{(x+1)^2} \cdot \left(\frac{(x+1)^2}{(x+1)^2 + 1}\right)^{3/2}$$

$$= \frac{x+1}{((x+1)^2 + 1)^{3/2}}$$

$$K\Big|_{x=2} = \frac{2+1}{((2+1)^2 + 1)^{3/2}}$$

$$= \frac{3}{10\sqrt{10}}$$

157. $r(t) = \langle 3\cos t, 3\sin t, t^2 \rangle$

$v = r'(t) = \langle -3\sin t, 3\cos t, 2t \rangle$

$a = r''(t) = \langle -3\cos t, -3\sin t, 2 \rangle$

$\|v(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$

$= \sqrt{9 + 4t^2}$

163. $r(t) = \langle t^2, 5t, t^2 - 16t \rangle$

$v(t) = \langle 2t, 5, 2t - 16 \rangle$

$\|v(t)\| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$

$= \sqrt{8t^2 - 64t + 281}$

$\|v(t)\|$ is a min when

$8t^2 - 64t + 281$ is a min

$(8t^2 - 64t + 281)' = 16t - 64$

critical point : $t = 4$

$(8t^2 - 64t + 281)'' = 16$

so $t = 4$ is a min

173. $V_0 = 500\,m/s$ $\theta = 60°$

time at max height

$s(t) = 500t\cos 60°\hat{i} + (500t\sin 60° - 4.9t^2)\hat{j}$

$= 250t\hat{i} + (250\sqrt{3}t - 4.9t^2)\hat{j}$

$v(t) = 250\hat{i} + (250\sqrt{3} - 9.8t)\hat{j}$

$250\sqrt{3} - 9.8t = 0$

$9.8t = 250\sqrt{3}$

$t = \frac{250\sqrt{3}}{9.8}$

$t = 44.185$ sec.

174. $V_0 = 500\,m/s$ $\theta = 60°$; max height

From problem 173

$s(t) = 250t\hat{i} + (250\sqrt{3}t - 4.9t^2)\hat{j}$

and $t = 44.185$ sec gives the time at max height

$s(44.185)$

$= 250(44.185)\hat{i} + [250\sqrt{3}(44.185) - 4.9(44.185)^2]\hat{j}$

$= 11046.242\hat{i} + 9566.327\hat{j}$

So max height is 9566.327m

175. $V_0 = 500\,m/s$, $\theta = 60°$

time at max range

From problem 173

$s(t) = 250t\hat{i} + (250\sqrt{3}t - 4.9t^2)\hat{j}$

$250\sqrt{3}t - 4.9t^2 = 0$

$t(250\sqrt{3} - 4.9t) = 0$

$t = 0$ $\quad t = \frac{250\sqrt{3}}{4.9}$

so $t = 88.37$ sec

176. $V_0 = 500 m/s$ $\theta = 60°$, max range

From problem 175
$S(t) = 250t\,\hat{i} + (250\sqrt{3}t - 4.9t^2)\hat{j}$
and $t = 88.37$ sec gives time at max range
$S(88.37)$
$= 250(88.37)\hat{i} + [250\sqrt{3}(88.37) - 4.9(88.37)^2]\hat{j}$
$= 22092.485\,\hat{i} - 0.026\,\hat{j}$
So the max range is
22092.485m

177. $V_0 = 500 m/s$, $\theta = 60°$, total flight time

total flight time would be the same as the time to the max range. So $t = 88.37$ sec

181. $\theta = 8°$; range $= 50m$

$\vec{R} = \frac{V_0^2 \sin 2\theta}{9.8}\hat{i}$

$50 = \frac{V^2 \sin 2(8)}{9.8}$

$490 = V^2 \sin 16°$

$V^2 = \frac{490}{\sin 16°}$

$V = \pm\sqrt{\frac{490}{\sin 16°}}$

$V = 42.16 m/s$

183. $a(t) = t\hat{j} + t\hat{k}$
$v(1) = 5\hat{j}$, $r(1) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$v(t) = c_1\hat{i} + (\frac{1}{2}t^2 + c_2)\hat{j} + (\frac{1}{2}t^2 + c_3)\hat{k}$
$v(1) = 5\hat{j}$
$c_1\hat{i} + (\frac{1}{2} + c_2)\hat{j} + (\frac{1}{2} + c_3)\hat{k} = 0\hat{i} + 5\hat{j} + 0\hat{k}$
$c_1 = 0$, $c_2 = 9/2$, $c_3 = -1/2$
$v(t) = 0\hat{i} + (\frac{1}{2}t^2 + \frac{9}{2})\hat{j} + (\frac{1}{2}t^2 - \frac{1}{2})\hat{k}$
$r(t) = b_1\hat{i} + (\frac{1}{6}t^3 + \frac{9}{2}t + b_2)\hat{j} + (\frac{1}{6}t^3 - \frac{1}{2}t + b_3)\hat{k}$
$r(1) = 0\hat{i} + 0\hat{j} + 0\hat{k}$
$b_1\hat{i} + (\frac{1}{6} + \frac{9}{2} + b_2)\hat{j} + (\frac{1}{6} - \frac{1}{2} + b_3)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$
$b_1 = 0$, $b_2 = -\frac{14}{3}$, $b_3 = \frac{1}{3}$
$r(t) = 0\hat{i} + (\frac{1}{6}t^3 + \frac{9}{2}t - \frac{14}{3})\hat{j} + (\frac{1}{6}t^3 - \frac{1}{2}t + \frac{1}{3})\hat{k}$

3. $V(x,y)=\pi x y$, $V(2,5)$

$V(2,5)=\pi(2)^2(5)$
$= 20\pi$

17. $g(x,y)=4-x-y$; $c=0,4$

$g(x,y)=0 \Rightarrow 4-x-y=0$
$4=x+y$ a line

$g(x,y)=4 \Rightarrow 4-x-y=4$
$x+y=0$ a line through the origin

21. $g(x,y)=\frac{x}{x+y}$, $c=-1,0,2$

$g(x,y)=-1 \Rightarrow \frac{x}{x+y}=-1$
$g(x,y)=0 \Rightarrow \frac{x}{x+y}=0$
$g(x,y)=2 \Rightarrow \frac{x}{x+y}=2$

49. $w(x,y,z)=x^2+y^2+z^2$; $c=9$

$x^2+y^2+z^2=9$
Sphere of radius 3

51. $w(x,y,z)=x^2+y^2-z^2$; $c=4$

$x^2+y^2-z^2=4$
hyperboloid of one sheet

56. $E=kl\sqrt{x^2+y^2}$; $E=10$, $E=100$, $k=1$

$E=10$: $10=l\sqrt{x^2+y^2}$
$E=100$: $100=l\sqrt{x^2+y^2}$

57. T is inversley proportional to the square of the distance from a point P to the origin

$T(x,y)=\frac{K}{x^2+y^2}$

58. $P(1,2)=50°$; $Q(3,4)$

from #57 $T(x,y)=\frac{K}{x^2+y^2}$
$50=\frac{K}{1^2+2^2}$
$50=K/5$
$250=K$
$T(3,4)=\frac{250}{3^2+4^2}$
$=\frac{250}{25}$
$T(3,4)=10$

59. $T \doteq 40°$, $T = 100°$

from 57: $T(x,y) = \frac{K}{x^2+y^2}$

$40 = \frac{K}{x^2+y^2}$

$x^2+y^2 = K/40$ (a circle w/ radius $\sqrt{10K}/20$)

$100 = \frac{K}{x^2+y^2}$

$x^2+y^2 = K/100$ (a circle w/ radius $\sqrt{K}/10$)

61. $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$

$= \frac{5(1)^2(2)}{1^2+2^2}$

$= \frac{10}{1+4}$

$= 2$

86. $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

a) along the x-axis

$\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

$= \lim_{x\to 0} \frac{x(0)+0^3}{x^2+0^2}$

$= \lim_{x\to 0} 0$

$= 0$

b) along the y-axis

$\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

$= \lim_{y\to 0} \frac{0(y)+y^3}{0^2+y^2}$

$= \lim_{y\to 0} y^3/y^2$

$= \lim_{y\to 0} y$

$= 0$

c) along $y = 2x$

$\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

$= \lim_{x\to 0} \frac{x(2x)+(2x)^3}{x^2+(2x)^2}$

$= \lim_{x\to 0} \frac{2x^2+8x^3}{x^2+4x^2}$

$= \lim_{x\to 0} \frac{2+8x}{1+4}$

$= \frac{2}{5}$

87. $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

this limit is DNE since along the path $y=2x$ the limit is different than the limits along the x and y axes.

88. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

a) along the x axis

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

$= \lim_{x\to 0} \frac{x^2(0)}{x^4+0}$

$= \lim_{x\to 0} 0$

$= 0.$

88 cont. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

b) along the y-axis

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$
$$= \lim_{y\to 0} \frac{0^2y}{0^4+y^2}$$
$$= \lim_{y\to 0} 0$$
$$= 0$$

c) along the path $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$
$$= \lim_{x\to 0} \frac{x^2(x^2)}{x^4+(x^2)^2}$$
$$= \lim_{x\to 0} \frac{x^4}{2x^4}$$
$$= \frac{1}{2}$$

89. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

this limit DNE
since along the path $y=x^2$ the limit is different than the limits along the x and y axes.